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An infinite plate with a curvilinear hole having two poles and arbitrary shape in the presence of heat

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Abstract– In this paper, we consider the boundary value problem for isotropic homogeneous perforated infinite elastic media in presence of uniform flow of heat. Then, we use the more general shape of conformal mapping to obtain the complex potential functions for the problem in the form integro-differential equation with singular kernel. Moreover, the three components of stress are calculated. Many special cases are obtained and several applications are discussed from the work. The results indicate that the effect of heat on an infinite plate with a curvilinear hole having two poles and arbitrary shape are very pronounced.

Keywords– Complex potential functions, curvilinear hole, conformal mapping, integro-differential equation, Goursat functions, stress components ,presence of heat.

1 Introduction and basic equations

Problems dealing with isotropic homogenous perforated infinite plate have been investigated by many authors, see [1-11]. The extensive literature on the topic is now available and we can only mention a few recent interesting investigations in [12–14]. The complex variable method has been applied to solve the first and second fundamental problems for $|\zeta|>1$, the infinite region outside a closed contour conformably mapped outside the unit circle γ with two poles . The first and second fundamental problem in the plane theory of elasticity are equivalent to finding analytic functions $\phi_1(z)$ and $\psi_1(z)$ of one complex argument $z=x+iy$.

These functions satisfying the boundary conditions

$$k\phi_1(t)-t\overline{\phi_1^{\prime}(t)}-\overline{\psi_1(t)}=f(t) \tag{1}$$

where $\phi_1(t)$ and $\psi_1(t)$ are two analytic functions, t denoting the affix of a point on the boundary. In the first fundamental boundary value problem $k=-1$, $f(t)$ is a given function of stresses ,while in the second fundamental boundary value problem $k=\chi$ and $f(t)$ is a given function of the displacement .

Let, the complex potentials $\phi_1(t)$ and $\psi_1(t)$ take the form

$$\phi_1(\zeta)=-\frac{X+iY}{2\pi(1+\chi)}\ln\zeta+c\Gamma\zeta+\phi(\zeta) \tag{2}$$

$$\psi_1(\zeta)=\chi\frac{(X-iY)}{2\pi(1+\chi)}\ln\zeta+c\Gamma^*\zeta+\psi(\zeta) \tag{3}$$

where X,Y are the components of the resultant vector of all external forces acting on the boundary and Γ, Γ^* are constants, generally complex functions $\phi(\zeta)$,$\psi(\zeta)$ are single-valued analytic functions within the region outside the unit circle γ and $\phi(\infty)=0,\psi(\infty)=0$.

Take the conformal mapping which mapped the domain of the curvilinear hole C on the domain outside a unit circle γ by the rational function

$$z=w(\zeta)\ , \quad |\zeta|<1,\ c>0\ . \tag{4}$$

And $w'(\zeta)$ dose not vanish or become infinite outside γ i.e.

$$w'(\zeta)\neq 0,\infty\ . \tag{5}$$

In [1] , Muskhelishvili used the rational mapping $z=c(\zeta+m\zeta^{-1})$, $c>0$,m is real number , (6)

to solve the problem of infinite plate weakened by an elliptic hole. El-Sirafy and Abdou in [2] . used the rational mapping ,

$$z=c\frac{\zeta+m\zeta^{-1}}{1-n\zeta^{-1}}\ ,\ |n|<1,\ c>0, \tag{7}$$

to solve the first and second fundamental problem for the infinite plate with general curvilinear C conformably mapped on the domain outside a unit circle .

The rational mapping

$$z=c\ (\zeta+m_1\zeta^{-1}+m_2\zeta^{-2})\ ,\quad 0\le m_1\le 1 \tag{8}$$

is used by Abdou and Khamis [3] are obtained the solution of the problem of an infinite plate with a curvilinear hole having three poles and shape and square shape as special cases of the hole .

Also, England [4] studied the conformal mapping function,

$$z=c\frac{\zeta+m\zeta^{-1}}{(1-n_1\zeta^{-1})\ (1-n_2\zeta^{-1})}\ ,\ c>0\ . \tag{9}$$

This study is useful for researchers who work on the studies of petroleum tubes industry or water or gas. It also benefits the physics scientists who work on the study of the ozone hole.

2 Formulation of the problem

Consider the rational mapping on the domain outside a unit circle γ by the rational function .

$$z = c\frac{(\zeta + m\zeta^{-1} + \ell\zeta^{-2})}{(1 - n\zeta^{-1})^2} \quad ,|n| < 1 \;,|\zeta| > 1 . \tag{10}$$

We can written the rational function in the form

$$z = c\frac{\zeta^3 + m\zeta + \ell}{(\zeta - n)^2} . \tag{11}$$

where m, n and ℓ are real number, $z'(\zeta)$ does not vanish or become infinite outside the unit circle γ. If a temperature distribution $\Theta = qy$ is following uniformly in the direction of the negative y-axis, where the increasing a temperature distribution Θ is assumed to be constant a cross the thickness of the plate, i.e. $\Theta = \Theta(x, y)$, and q is the constant temperature gradient. The uniform flow of heat is distributed by the presence of an insulated curvilinear hole C. The heat equation satisfies the relation, see Fig. 1 .

$$\nabla^2\Theta = 0 \;, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{12}$$

$$\frac{\partial\Theta}{\partial n} = 0 \quad , \quad r = r_0 \tag{13}$$

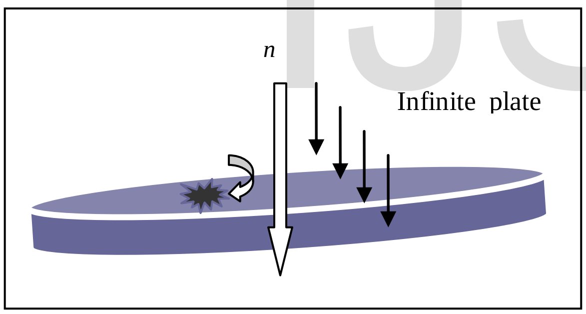

Fig. 1: The uniform flow of heat Θ curvilinear region

where n is the unit vector perpendicular to the surface.

Neglecting the variation of the strain and the stress with respect to the thickness of the plate, the thermoelastic potential Φ satisfies the formula, see Parkus [5].

$$\nabla^2\Phi = (1 + \nu)\; \alpha \;\Theta \tag{14}$$

where α is a scalar which present the coefficient of the thermal expansion and is Poisson's ratio ν. Assume the force of the plate is free of applied loads.

In this case, the formula (1) for the first and second boundary value problems respectively take the following form,

$$\phi_1(t) + t\overline{\phi_1^{\backslash}}(t) + \overline{\psi_1}(t) = \frac{\partial\Phi}{\partial x} + i\frac{\partial\Phi}{\partial y} + \frac{1}{2G}\int_0^s [iX(s) - Y(s)]\; ds + c \tag{15}$$

$$k\phi_1(t) - t\overline{\phi_1^{\backslash}}(t) - \overline{\psi_1}(t) = u + iv - \frac{\partial\Phi}{\partial x} - i\frac{\partial\Phi}{\partial y} \tag{16}$$

where the applied stresses *X(s)* and *Y(s)* are prescribed on the boundary of the plane *s* is the length measured from an arbitrary point, *u* and *v* are the displacement components , *G* is the shear modulus. Also, here the applied stresses *X(s)* and *Y(s)* are must satisfy the following , see Parkus [5].

$$X(s) = \sigma_{xx}\frac{dy}{ds} - \sigma_{xy}\frac{dx}{ds} \tag{17}$$

$$Y(s) = \sigma_{yx}\frac{dy}{ds} - \sigma_{yy}\frac{dx}{ds} \tag{18}$$

where σ_{xx} , σ_{yy} and σ_{yx} are the components of stresses which are given by the following relations

$$\sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} = 2G\;[\frac{\partial^2\Phi}{\partial y^2} - \frac{\partial^2\Phi}{\partial x^2} + 2i\frac{\partial^2\Phi}{\partial x\partial y}] + 4G\{\overline{z}\;\phi''(z) + \psi''(z)\} \tag{19}$$

$$\sigma_{xx} + \sigma_{yy} = 4G\;[\mathrm{Re}\,\phi'(z) - \lambda\Theta] \tag{20}$$

where $\lambda = \frac{\alpha}{2}(1 + \nu)$ is the coefficient of heat transfer.

3 The rational mapping

The mapping function (11) maps the curvilinear hole C where the origin lies outside the hole under the conditions that $w'(\zeta)$ dose not vanish or become infinite outside the unit circle γ. The following graphs give the different shapes of the rational mapping (11), see Fig. 2 .

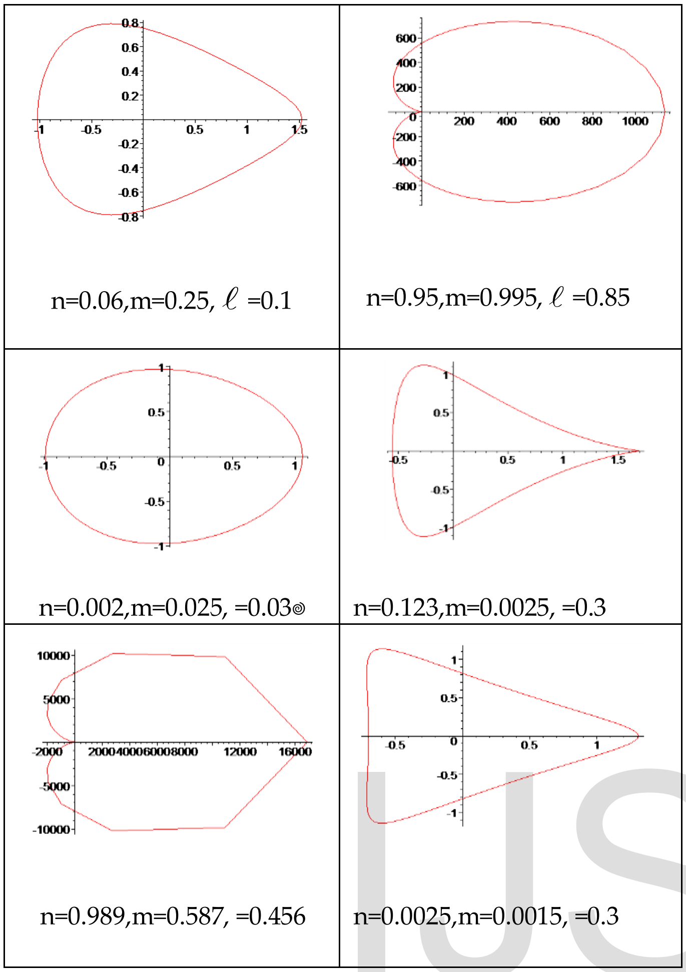

Fig. 2: The different shapes of the rational mapping (11)

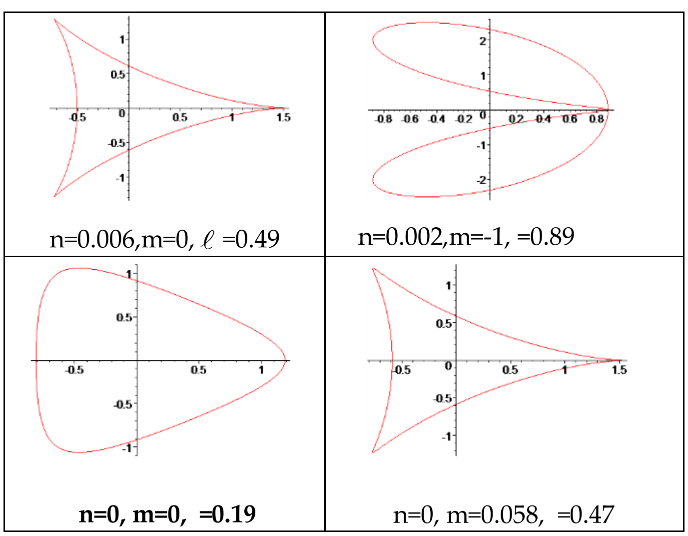

Fig. 3 : The different shapes of the rational mapping for special cases

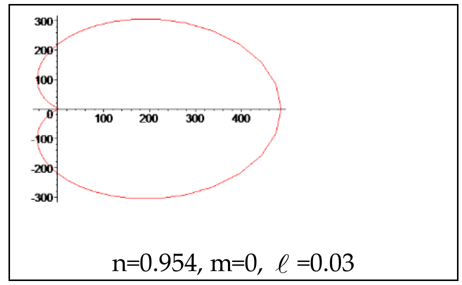

Fig. 3: The different shapes of the rational mapping for special cases

4 Method of Solution

In this section ,we use the complex variable method to obtain the two complex functions (Goursat functions) $\phi(\zeta)$,$\psi(\zeta)$. Moreover, the three stress components σ_{xx}, σ_{yy} and σ_{yx} will be completely determined .

(i) The Components of Stresses:

The solution of Eq. (12) is given by,

$$\Theta = q[R + \frac{r_0^2}{R}],\ \ R = \sqrt{x^2 + y^2} \tag{21}$$

By substitute Eq.(21) in Eq.(12) and using the definition of $\nabla^2\Phi$ In polar coordinates the thermoelastic potential function take the form,

$$\Phi = \frac{(1+\nu)\alpha q r_0^2}{4} \ln z\ R^2 \tag{22}$$

Also, the stresses components can be adapted in the form,

$$\sigma_{xx} = 2G\left[-\frac{1}{2}\left(\frac{\partial^2\Phi}{\partial y^2} - \frac{\partial^2\Phi}{\partial x^2} + 2\lambda\Theta\right) + \mathrm{Re}\{2\phi'(z) - \bar{z}\,\phi''(z) - \psi''(z)\} \right] \tag{23}$$

$$\sigma_{yy} = 2G\left[\frac{1}{2}\left(\frac{\partial^2\Phi}{\partial y^2} - \frac{\partial^2\Phi}{\partial x^2} - 2\lambda\Theta\right) + \mathrm{Re}\{2\phi'(z) + \bar{z}\,\phi''(z) + \psi''(z)\} \right] \tag{24}$$

and

$$\sigma_{xy} = 2G\left[\frac{\partial^2\Phi}{\partial x \partial y} + \mathrm{Im}\{\bar{z}\,\phi''(z) + \psi''(z)\} \right] \tag{25}$$

Eqs.(23) , (24) and (25) after some derivatives and al-gebraic relations adapted as,

$$\sigma_{xx} = 2G\left[-\eta(z^2 + 4z\bar{z} + \bar{z}^2)\,\mathrm{Im}\,z + \mathrm{Re}\{2\phi'(z) - M(z,\bar{z})\} \right] \tag{26}$$

$$\sigma_{yy} = 2G\left[\eta(z^2 + 4z\bar{z} + \bar{z}^2)\,\mathrm{Im}\,z + \mathrm{Re}\{2\phi'(z) + M(z,\bar{z})\} \right] \tag{27}$$

and

$$\sigma_{xy} = 2G\left[\eta(z\bar{z} - 2(\operatorname{Im} z)^2)\operatorname{Re} z + \operatorname{Im} M(z,\bar{z})\right] \qquad (28)$$

where

$$\eta = \frac{(1+\nu)r_0^2}{2(z\bar{z})^2}, \quad M(z,\bar{z}) = \bar{z}\,\phi''(z) + \psi'(z). \qquad (29)$$

(ii) Goursat functions:

To obtain the two complex potential functions (Goursat functions) by using the conformal mapping (11) in the boundary condition (5).We write the expression $\dfrac{w(\zeta)}{w'(\zeta^{-1})}$ in the form ,

$$\frac{w(\zeta)}{w'(\zeta^{-1})} = \alpha(\zeta) + \beta(\zeta^{-1}) \qquad (30)$$

where,

$$\alpha(\zeta) = \frac{h}{(\zeta - n)^2} \qquad (31)$$

$\beta(\zeta^{-1})$ is a regular function for $|\zeta| > 1$, $\dfrac{w(\zeta)}{w'(\zeta^{-1})}$ has a singularity at $\zeta = n$, and

$$\begin{aligned} h = (1-n^2)^3 S^{-2}(n)[-3S(n)(n^4 + mn^2 + \ell n)(1-n^2)^{-1} \\ + S(n)(3n^2 + m) - S'(n)(n^3 + mn + \ell)] \end{aligned} \qquad (32)$$

where

$$S(\sigma) = 1 - 3n\sigma - m\sigma^2 - (mn + 2\ell)\,\sigma^3 \qquad (33)$$

Using Eq.(2) and Eq.(3) in Eq.(1), we get

$$k\phi(\sigma) - \alpha(\sigma)\,\overline{\phi'(\sigma)} - \overline{\psi_*(\sigma)} = G(\sigma) \qquad (34)$$

where

$$\psi_*(\sigma) = \psi(\sigma) + \beta(\sigma)\,\phi'(\sigma) \qquad (35)$$

$$\begin{aligned} G(\sigma) = F(\sigma) - ck\Gamma\sigma + \frac{c\overline{\Gamma^*}}{\sigma} + N(\sigma)\,\alpha(\sigma) \\ + N(\sigma)\,\overline{\beta(\sigma)} \end{aligned} \qquad (36)$$

$$N(\sigma) = \left[c\bar{\Gamma} - \frac{\sigma(X - iY)}{2\pi(1+\chi)}\right], \qquad F(\sigma) = f(t) \qquad (37)$$

Assume that the function $F(\sigma)$ with its derivatives must satisfy the Holder condition. Our aim is to determine the functions $\phi(\zeta)$ and $\psi(\zeta)$ for the various boundary value problems. For this multiply both sides of Eq. (34) by $\dfrac{d\sigma}{2\pi i(\sigma - \zeta)}$ where ζ is any point in the interior of γ and integral over the circle , we obtain

$$\begin{aligned} \frac{k}{2\pi i}\int_\gamma \frac{\phi(\sigma)}{\sigma - \zeta}\,d\sigma - \frac{1}{2\pi i}\int_\gamma \frac{\alpha(\sigma)\overline{\phi'(\sigma)}}{\sigma - \zeta}\,d\sigma - \\ \frac{1}{2\pi i}\int_\gamma \frac{\overline{\psi_*(\sigma)}}{\sigma - \zeta}\,d\sigma = \frac{1}{2\pi i}\int_\gamma \frac{G(\sigma)}{\sigma - \zeta}\,d \end{aligned} \qquad (38)$$

Using Eqs.(36)-(37) in Eq.(38) then applying the properties of Cauchy integral , to have

$$\frac{k}{2\pi i}\int_\gamma \frac{\phi(\sigma)}{\sigma - \zeta}\,d\sigma = -k\phi(\zeta) \qquad (39)$$

and

$$\frac{1}{2\pi i}\int_\gamma \frac{\alpha(\sigma)\overline{\phi'(\sigma)}}{\sigma - \zeta}\,d\sigma = ch\sum_{i=1}^{2}\frac{b_i}{(n-\zeta)^i} \qquad (40)$$

where b_1 and b_2 are complex constants which can be determined.

$$\frac{1}{2\pi i}\int_\gamma \frac{N(\sigma)\,\alpha(\sigma)}{(\sigma - \zeta)}\,d\sigma = -\frac{h\,(X - iY)}{2\pi(1+\chi)(n-\zeta)} - \frac{N(n)\,h}{(n-\zeta)^2} \qquad (41)$$

Also,

$$\begin{aligned} \frac{1}{2\pi i}\int_\gamma \frac{G(\sigma)}{(\sigma - \zeta)}\,d\sigma = A(\zeta) - \frac{c\overline{\Gamma^*}}{\zeta} \\ - \frac{N(n)\,h}{(n-\zeta)^2} - \frac{h\,(X - iY)}{2\pi(1+\chi)(n-\zeta)} \end{aligned} \qquad (42)$$

where,

$$A(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{F(\sigma)}{(\sigma - \zeta)}\,d\sigma,\ N(\sigma) = \left[c\bar{\Gamma} - \frac{\sigma(X - iY)}{2\pi(1+\chi)}\right] \qquad (43)$$

From the above ,Eq.(38) becomes

$$\begin{aligned} -k\phi(\zeta) = A(\zeta) - \frac{c\overline{\Gamma^*}}{\zeta} - \frac{h\,(X - iY)}{2\pi(1+\chi)(n-\zeta)} \\ - \frac{N(n)\,h}{(n-\zeta)^2} + ch\sum_{i=1}^{2}\frac{b_i}{(n-\zeta)^i} \end{aligned} \qquad (44)$$

To determined b_i , $i = 1,2$ differentiating Eq.(44) with respect to ζ and substituting in Eq.(40) , we get

$$\begin{aligned} \frac{1}{2\pi i}\int_\gamma \frac{h}{(\sigma - \zeta)(\sigma - n)^2}\Big[-\overline{A'(\sigma)} - c\Gamma^*\sigma^2 \\ + \frac{2h\sigma^3\overline{N(n)}}{(n\sigma - 1)^3} + \frac{h\sigma^2(X + iY)}{2\pi(1+\chi)(n\sigma - 1)^2} \\ - ch\sum_{i=1}^{2}\frac{\sigma^i\,\overline{b_i}}{(n\sigma - 1)^i}\Big] = ckh\sum_{i=1}^{2}\frac{b_i}{(n-\zeta)^i} \end{aligned} \qquad (45)$$

Substituting Eq.(31) in Eq.(45), then using the properties of Cauchy integral and applying the reside theorem at the singular points ,we obtain

$$\overline{A'(n)} + c\Gamma^* n^2 - 2\upsilon_1^3 h\ \overline{N(n)} - \frac{h\upsilon_2(X+iY)}{2\pi(1+\chi)} + ch\sum_{i=1}^{2}(\upsilon_i \overline{b_i}) + ck\sum_{i=1}^{2} b_i = 0 \tag{46}$$

$$\upsilon_i = \frac{n^i}{(n^2-1)^i},\ i = 1,2. \tag{47}$$

The last equation can be written in the form

$$c\ \sum_{i=1}^{2}(kb_i + \upsilon_i h\bar{b}_i) = E \tag{48}$$

where,

$$E = -\overline{A'(n)} - c\Gamma^* n^2 + 2\upsilon_1^3 h\ \overline{N(n)} + \frac{h\upsilon_2(X+iY)}{2\pi(1+\chi)} \tag{49}$$

taking the complex conjugate of Eq.(48), we get

$$c\ \sum_{i=1}^{2}(k\bar{b}_i + \upsilon_i h b_i) = \bar{E} \tag{50}$$

form Eq.(48) and (50) , we have

$$b_i = \frac{kE - \upsilon_i h\bar{E}}{c(k^2 - \upsilon_i^2 h^2)},\ i = 1,2. \tag{51}$$

To obtain the complex function $\psi(\zeta)$ we have from Eq.(35) after substituting the expression of $\psi_*(\sigma)$ and $G(\sigma)$, and taking the complex conjugate of the resulting equation after using the expression of $\overline{\beta(\sigma)}$ to yields,

$$\psi(\sigma) = -\overline{F(\sigma)} + ck\bar{\Gamma}\sigma^{-1} - c\Gamma^*\sigma + k\overline{\phi(\sigma)} - \overline{\alpha(\sigma)}\ \phi_*(\sigma) - \frac{\overline{w(\sigma)}}{w'(\sigma)}\phi_*(\sigma) + \frac{h\sigma^2}{(1-n\sigma)^2}\phi_*(\sigma) \tag{52}$$

where,

$$\phi_*(\sigma) = \phi'(\sigma) + \overline{N(\sigma)} \quad ,\overline{N(\sigma)} = \left[c\Gamma - \frac{\sigma^{-1}(X+iY)}{2\pi\ (1+\chi)}\right] \tag{53}$$

and calculate sum residue, we obtain Multiplying both sides of Eq.(52) by $\frac{1}{2\pi i(\sigma-\zeta)}$, where ζ is any point in the interior of γ and integrating over the circle ,then using the properties of Cauchy's integral and calculating the sum residue ,we obtain

$$\psi(\zeta) = ck\bar{\Gamma}\zeta^{-1} - \frac{\overline{w(\zeta)}}{w'(\zeta)}\phi_*(\zeta) + \frac{h\zeta^2}{(1-n\zeta)^2}\phi_*(n^{-1}) + B(\zeta) - B \tag{54}$$

where,

$$B(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{\overline{F(\sigma)}}{(\sigma-\zeta)}\,d\sigma \tag{55}$$

and

$$B = \frac{1}{2\pi i}\int_\gamma \frac{\overline{F(\sigma)}}{\sigma}\,d\sigma \tag{56}$$

5 Special cases

Now, we will offer some cases:

i. Let $m = 0,\ n \neq 0$, the edge of the hole resembles the shape of a heart, see Fig. 3

$$z = cw(\zeta) = c\frac{\zeta^3 + \ell}{(\zeta - n)^2} \tag{57}$$

$$h = M^{-1}(n)[3n^2(1-n^2)^3 - 3n(1-n^2)^2(n^3+\ell)] - M^{-2}(n)[(n^3+\ell)(1-n^2)^3(-3n-6\ell n^2)] \tag{58}$$

where

$$M(\sigma) = 1 - 3n\sigma - 2\ell\sigma^3 \tag{59}$$

Then (44) and (54) becomes

$$-k\phi(\zeta) = A(\zeta) - \frac{c\overline{\Gamma^*}}{\zeta} - \frac{N(n)\ h}{(n-\zeta)^2} - \frac{h\ (X-iY)}{2\pi(1+\chi)(n-\zeta)} + ch\ \sum_{i=1}^{2}\frac{b_i}{(n-\zeta)^i} \tag{60}$$

and

$$\psi(\zeta) = ck\bar{\Gamma}\zeta^{-1} - \frac{\overline{w(\zeta)}}{w'(\zeta)}\phi_*(\zeta) + \frac{h\zeta^2}{(1-n\zeta)^2}\phi_*(n^{-1}) + B(\zeta) - B \tag{61}$$

where

$$E = -\overline{A'(n)} - c\Gamma^* n^2 + 2\upsilon_1^3 h\ \overline{N(n)} + \frac{h\upsilon_2(X+iY)}{2\pi(1+\chi)} \tag{62}$$

$$b_i = \frac{kE - \upsilon_i h\bar{E}}{c(k^2 - \upsilon_i^2 h^2)},\ i = 1,2. \tag{63}$$

ii. For $n = 0,\ 0 \le m \le 1$, we get the mapping function represented to the hole which is the dart, see Fig. 3

$$z = c\frac{\zeta^3 + m\zeta + \ell}{\zeta^2} \tag{64}$$

$$h = L^{-1}(n)(3n^2 + m) - L^{-2}(n) .[(n^3 + mn + \ell)(-2mn - 6\ell n^2)] \tag{65}$$

where

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$$L(\sigma)=1-m\sigma^2-2\ell\sigma^3. \tag{66}$$

Then (44) and (54) becomes

$$-k\phi(\zeta)=A(\zeta)-\frac{c\overline{\Gamma^*}}{\zeta}-\frac{N(n)\ h}{(n-\zeta)^2}-\frac{h\ (X-iY)}{2\pi(1+\chi)(n-\zeta)}+ch\sum_{i=1}^{2}\frac{b_i}{(n-\zeta)^i} \tag{67}$$

and

$$\psi(\zeta)=ck\overline{\Gamma}\zeta^{-1}-\frac{\overline{w(\zeta)}}{w'(\zeta)}\phi_*(\zeta)+\frac{h\zeta^2}{(1-n\zeta)^2}\phi_*(n^{-1})+B(\zeta)-B \tag{68}$$

where

$$E=-\overline{A'(n)}-c\Gamma^*n^2+2\upsilon_1^3h\ \overline{N(n)}+\frac{h\upsilon_2(X+iY)}{2\pi(1+\chi)}$$

iii. Let $m=n=0$, we get the mapping function represented to the hole, see Fig. 3

$$z=c\ \frac{\zeta^3+\ell}{\zeta^2} \tag{69}$$

$$h=\frac{n^2(6\ell^2+3)}{(1-2\ell n^3)^2} \tag{70}$$

Then (44) and (54) becomes

$$-k\phi(\zeta)=A(\zeta)-\frac{c\overline{\Gamma^*}}{\zeta}-\frac{N(n)\ h}{(n-\zeta)^2}-\frac{h\ (X-iY)}{2\pi(1+\chi)(n-\zeta)}+ch\sum_{i=1}^{2}\frac{b_i}{(n-\zeta)^i} \tag{71}$$

And

$$\psi(\zeta)=ck\overline{\Gamma}\zeta^{-1}-\frac{\overline{w(\zeta)}}{w'(\zeta)}\phi_*(\zeta)+\frac{h\zeta^2}{(1-n\zeta)^2}\phi_*(n^{-1})+B(\zeta)-B \tag{72}$$

where

$$E=-\overline{A'(n)}-c\Gamma^*n^2+2\upsilon_1^3h\ \overline{N(n)}+\frac{h\upsilon_2(X+iY)}{2\pi(1+\chi)}$$

iv. Let $m=-1$, we the edge of the hole resembles the shape of a shell, see Fig. 3

$$z=c\frac{\zeta^3-\zeta+\ell}{(\zeta-n)^2} \tag{73}$$

$$h=S^{-1}(n)[-3n\ \ (1-n^2)^2(n^3-n+\ell)+(3n^2-1)(1-n^2)^3]-S^{-2}(n)[(n^3-n+\ell).(1-n^2)^3S'(n)] \tag{74}$$

where

$$S(\sigma)=1-3n\sigma+2\sigma^2+(n-2\ell)\sigma^3. \tag{75}$$

Then (44) and (54) becomes

$$-k\phi(\zeta)=A(\zeta)-\frac{c\overline{\Gamma^*}}{\zeta}-\frac{N(n)\ h}{(n-\zeta)^2}-\frac{h\ (X-iY)}{2\pi(1+\chi)(n-\zeta)}+ch\sum_{i=1}^{2}\frac{b_i}{(n-\zeta)^i} \tag{76}$$

$$\psi(\zeta)=ck\overline{\Gamma}\zeta^{-1}-\frac{\overline{w(\zeta)}}{w'(\zeta)}\phi_*(\zeta)+\frac{h\zeta^2}{(1-n\zeta)^2}\phi_*(n^{-1})+B(\zeta)-B \tag{77}$$

where

$$E=-\overline{A'(n)}-c\Gamma^*n^2+2\upsilon_1^3h\ \overline{N(n)}+\frac{h\upsilon_2(X+iY)}{2\pi(1+\chi)}$$

v. Let $m=-n^2$, we get the mapping function represented to the hole which is the dart, see Fig. 3

$$z=c\frac{\zeta^3-n^2\zeta+\ell}{(\zeta-n)^2} \tag{78}$$

$$h=J^{-1}(n)[-3n\ell\ \ (1-n^2)^2+2n^2(1-n^2)^3]-J^{-2}(n)[\ell\ \ (1-n^2)^3J'(n)] \tag{79}$$

where

$$J(\sigma)=1-3n\sigma+n^2\sigma^2+(n^3-2\ell)\sigma^3 \tag{80}$$

Then (44) and (54) becomes

$$-k\phi(\zeta)=A(\zeta)-\frac{c\overline{\Gamma^*}}{\zeta}-\frac{N(n)\ h}{(n-\zeta)^2}-\frac{h\ (X-iY)}{2\pi(1+\chi)(n-\zeta)}+ch\sum_{i=1}^{2}\frac{b_i}{(n-\zeta)^i} \tag{81}$$

$$\psi(\zeta)=ck\overline{\Gamma}\zeta^{-1}-\frac{\overline{w(\zeta)}}{w'(\zeta)}\phi_*(\zeta)+\frac{h\zeta^2}{(1-n\zeta)^2}\phi_*(n^{-1})+B(\zeta)-B \tag{82}$$

where

$$E=-\overline{A'(n)}-c\Gamma^*n^2+2\upsilon_1^3h\ \overline{N(n)}+\frac{h\upsilon_2(X+iY)}{2\pi(1+\chi)}$$

6 Some applications

In this section, we assume different values of the given functions in the first or second fundamental boundary value problems. Then, we obtain the expression of Goursat functions. After that, the components of stresses can be calculated directly.

Curvilinear hole for an infinite layer subjected to uniform tensile stress and flowing heat.

(1) For $k=-1$, $\Gamma=\dfrac{p}{4}$, $\Gamma^*=-\dfrac{1}{2}p\ e^{-2i\theta}$ and $X=Y=f=0$, $0\le\theta\le 2\pi$, we have the case of infinite plate stretched at infinity by the application of a uniform tensile stress of intensity p, making an angle θ with the x-axis. The plate weakened by the curvilinear hole C which is free from stresses.

Then the functions in (44) and (54) becomes

$$f=0\Rightarrow A(\zeta)=0 \qquad (83)$$

$$N(n)=\left[c\overline{\Gamma}-\frac{n(X-iY)}{2\pi(1+\chi)}\right]=\frac{c\ p}{4} \qquad (84)$$

$$E=\frac{c\ n^2 p\ e^{-2i\theta}+\upsilon_1^3 h\ c\ p}{2},\ \overline{E}=\frac{c\ n^2 p\ e^{2i\theta}+\upsilon_1^3 h\ c\ p}{2} \qquad (85)$$

$$cb_i=\frac{-E-\upsilon_i h\overline{E}}{(1-\upsilon_i^2 h^2)}\quad ,i=1,2 \qquad (86)$$

$$\phi(\zeta)=\frac{c\ p\ e^{2i\theta}}{2\zeta}-\frac{c\ h\ p}{4(n-\zeta)^2}+c\ h\sum_{i=1}\frac{b_i}{(n-\zeta)^i} \qquad (87)$$

$$\psi(\zeta)=-\frac{c\ p}{4\zeta}-\frac{w(\zeta^{-1})}{w'(\zeta)}\phi_*(\zeta)+\frac{h\ \zeta^2}{(1-n\zeta)^2}\phi_*(n^{-1}) \qquad (88)$$

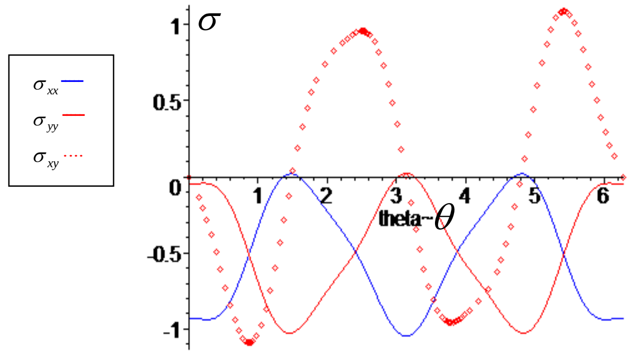
where

$$\phi_*(\zeta)=\phi'(\zeta)+\frac{c\ p}{4}.$$

For $(n=0.002, m=0.025, \ell=0.03, c=2, p=0.25)$ the stress components σ_{xx}, σ_{yy} and σ_{xy} are obtained in large forms calculated by computer and illustrated in following two cases:

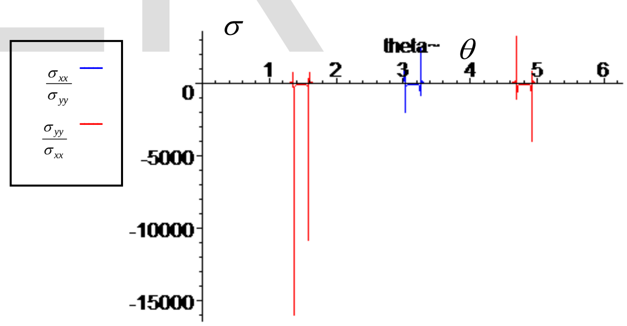
(i) When the study in the normal plate, we have the following shapes for the stress components, see Figs.(4-5)

(ii) In the thetmoelasticity plate, we have the following shapes for the stress components by using the substitutions $G=0.5$, $q=0.1$, $r_0=0.75$, $\alpha=0.7$, $\nu=1$ see Figs.(6-7).



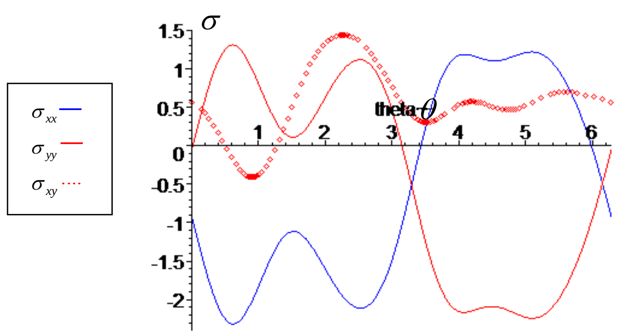
σ_{xx} has positive values at $0.42675\pi \le \theta \le 0.50955\pi$ and $1.49045\pi \le \theta \le 1.56369\pi$, σ_{yy} has positive values at $0.94904\pi \le \theta \le 1.03503\pi$, σ_{xy} has positive values at $0.46815\pi \le \theta \le 1.00318\pi$ and $1.52548\pi \le \theta \le 1.99682\pi$

Fig. 4: The relation between components of stresses and the angle made on the x-axis in the normal plate.



$\max\dfrac{\sigma_{xx}}{\sigma_{yy}}$ at $\theta\approx 1.03503\pi$, $\max\dfrac{\sigma_{yy}}{\sigma_{xx}}$ at $\theta\approx 1.49044\pi$

Fig. 5: The ratio of $\dfrac{\sigma_{xx}}{\sigma_{yy}}$ and $\dfrac{\sigma_{yy}}{\sigma_{xx}}$ in the normal plate



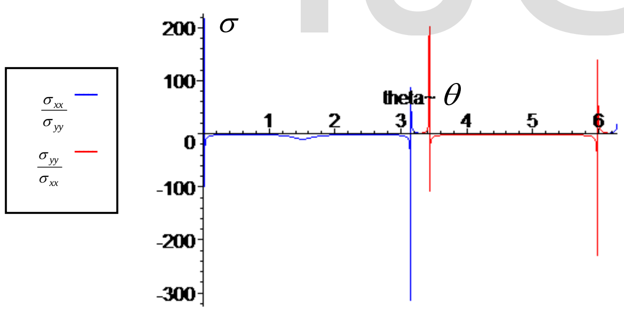
σ_{xx} has positive values at $1.08917\pi \le \theta \le 1.89808\pi$

σ_{yy} has positive values at $0 \le \theta \le 0.99682\pi$

σ_{xy} has positive values at $0 \le \theta \le 0.15286\pi$ and

$0.40445\pi \le \theta \le 2\pi$

Fig. 6: The relation between components of stresses and the angle made on the x-axis in the thermoelasticity plate.



$\max \frac{\sigma_{xx}}{\sigma_{yy}}$ at $\theta \approx 0$, $\max \frac{\sigma_{yy}}{\sigma_{xx}}$ at $\theta \approx 1.09235\pi$

Fig. 7: The ratio of $\frac{\sigma_{xx}}{\sigma_{yy}}$ and $\frac{\sigma_{yy}}{\sigma_{xx}}$ in the thermoelasticity plate

(2) For $k=-1$, $\Gamma=\Gamma^*=X=Y=0$ and $f=Pt$, where P is a real constant .Then the functions in (44) and (54) becomes

$$f=Pt \Rightarrow f=\frac{P\,c\,(\sigma^3+m\sigma+\ell)}{(\sigma-n)^2} \tag{89}$$

$$\overline{f}=\frac{c\,P\,(1+m\sigma^2+\ell\sigma^3)}{\sigma\,(1-n\sigma)^2} \tag{90}$$

$$N(n)=0 \tag{91}$$

$$A(\zeta)=cp\,\frac{(n-\zeta)\,(3n^2+m)-(n^3+m\,n+\ell)}{(n-\zeta)^2} \tag{92}$$

$$\overline{A'(n)}=cp\,\frac{n^2}{(1-n^2)^2}[2(3n^2+m)-2(1-n^2)\cdot(n^4+m\,n^2+\ell n)-(3n^2+m)] \tag{93}$$

$$E=\overline{E}=-\overline{A'(n)}=-cp\,\frac{n^2}{(1-n^2)^2}[2(3n^2+m)-2(1-n^2)\cdot(n^4+m\,n^2+\ell n)-(3n^2+m)] \tag{94}$$

$$cb_i=\frac{-E-\upsilon_i h\overline{E}}{(1-\upsilon_i^2 h^2)} \quad ,i=1,2 \tag{95}$$

$$\phi(\zeta)=cp\frac{(n-\zeta)(3n^2+m)-(n^3+m\,n+\ell)}{(n-\zeta)^2}+\sum_{i=1}^{2}\frac{c\,h\,b_i}{(n-\zeta)^i} \tag{96}$$

$$\psi(\zeta)=-cP(2n+\frac{1}{\zeta})-\frac{w(\zeta^{-1})}{w'(\zeta)}\phi'(\zeta)+\frac{h\,\zeta^2}{(1-n\zeta)^2}\phi'(n^{-1}) \tag{97}$$

where

$$B(\zeta)=\frac{1}{2\pi i}\int_\gamma\frac{\overline{F(\sigma)}}{(\sigma-\zeta)}\,d\sigma$$

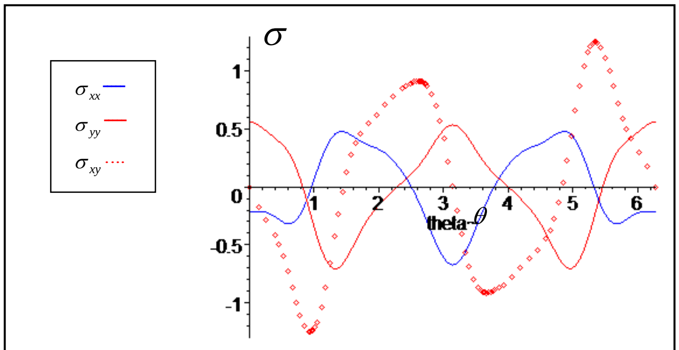
$$=\frac{c\,P}{2\pi i}\int_\gamma\frac{(1+m\sigma^2+\ell\sigma^3)}{\sigma\,(1-n\sigma)^2(\sigma-\zeta)}\,d\sigma=-\frac{c\,P}{\zeta}$$

$$B=\frac{c\,P}{2\pi i}\int_\gamma\frac{(1+m\sigma^2+\ell\sigma^3)}{\sigma^2(1-n\sigma)^2}\,d\sigma=2c\,P\,n$$

For ($n=0.002$, $m=0.025$, $\ell=0.03$, $c=2$, $p=0.25$), the stress components σ_{xx}, σ_{yy} and σ_{xy} are obtained in large forms calculated by computer and illustrated in following two cases:

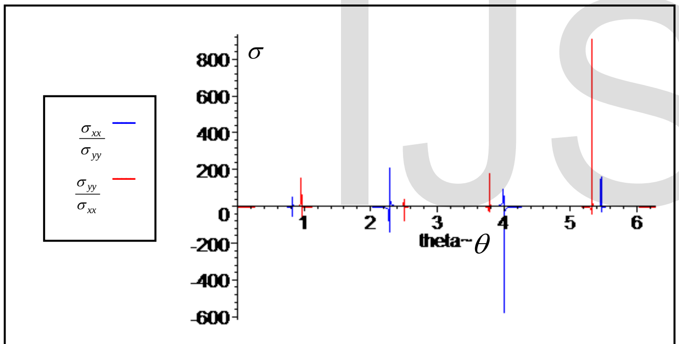
(i) When the study in the normal plate, we have the following shapes for the stress components, see Figs.(8-9)

(ii) In the thetmoelasticity plate, we have the following shapes for the stress components by using the substitutions G = 0.5, q =0.1 , r_0 = 0.75, α = 0.7, $\nu=1$ see Figs.(10-11).



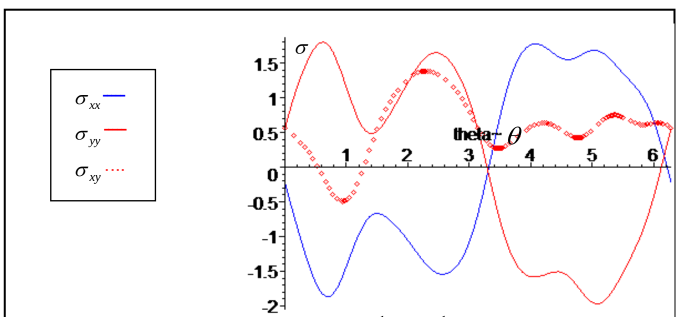
σ_{xx} has positive values at $0.30255\,\pi \le \theta \le 0.79617\,\pi$ and $1.20382\,\pi \le \theta \le 1.69108\,\pi$, σ_{yy} has positive values at $0 \le \theta \le 0.25477\,\pi$, $0.71975\,\pi \le \theta \le 1.27708\,\pi$ and $1.73248\,\pi \le \theta \le 2\,\pi$, σ_{xy} has positive values at $0.45223\,\pi \le \theta \le 0.99682\,\pi$ and $1.53503\,\pi \le \theta \le 2\,\pi$.

Fig. 8: The relation between components of stresses and the angle made on the x-axis in the normal plate.



max $\frac{\sigma_{xx}}{\sigma_{yy}}$ at $\theta \approx 0.72611\,\pi$, max $\frac{\sigma_{yy}}{\sigma_{xx}}$ at $\theta \approx 1.69745\,\pi$

Fig. 9: The ratio of $\frac{\sigma_{xx}}{\sigma_{yy}}$ and $\frac{\sigma_{yy}}{\sigma_{xx}}$ in the normal plate

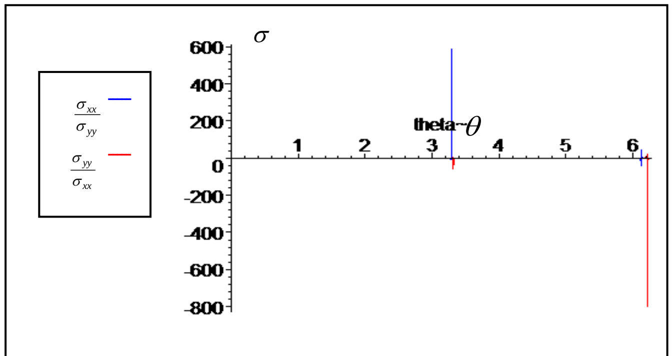


σ_{xx} has positive values at $1.0414\,\pi \le \theta \le 1.97134\,\pi$

σ_{yy} has positive values at $0 \le \theta \le 1.0414\,\pi$

σ_{xy} has positive values at $0 \le \theta \le 0.16242\,\pi$ and $0.40445\,\pi \le \theta \le 2\,\pi$.

Fig. 10 : The relation between components of stresses and the angle made on the x-axis in the thermoelasticity plate.



max $\frac{\sigma_{xx}}{\sigma_{yy}}$ at $\theta \approx 1.04458\,\pi$, max $\frac{\sigma_{yy}}{\sigma_{xx}}$ at $\theta \approx 1.97134\,\pi$

Fig. 11: The ratio of $\frac{\sigma_{xx}}{\sigma_{yy}}$ and $\frac{\sigma_{yy}}{\sigma_{xx}}$ in the thermoelasticity plate.

(3) For $k=\chi$, $\Gamma=\Gamma^*=f=0$, Then the functions in (44) and (54) becomes

$$f=0 \Rightarrow A(\zeta)=0 \qquad (98)$$

$$N(n)=-\frac{n\,(X-iY)}{2\pi(1+\chi)} \qquad (99)$$

$$E=\frac{h\,(X+iY)[\upsilon_2-2\,n\,\upsilon_1{}^3]}{2\pi(1+\chi)} \qquad (100)$$

$$\overline{E}=\frac{h\,(X-iY)[\upsilon_2-2\,n\,\upsilon_1{}^3]}{2\pi(1+\chi)} \qquad (101)$$

$$cb_i=\frac{\chi\,E-\upsilon_i h\overline{E}}{(\chi^2-\upsilon_i{}^2h^2)}\quad, i=1,2 \qquad (102)$$

$$\phi(\zeta)=\frac{h\,(X-iY)}{2\chi\pi(1+\chi)(n-\zeta)}\left[1-\frac{n}{(n-\zeta)}\right]-ch\sum_{i=1}^{2}\frac{b_i}{\chi(n-\zeta)^i} \qquad (103)$$

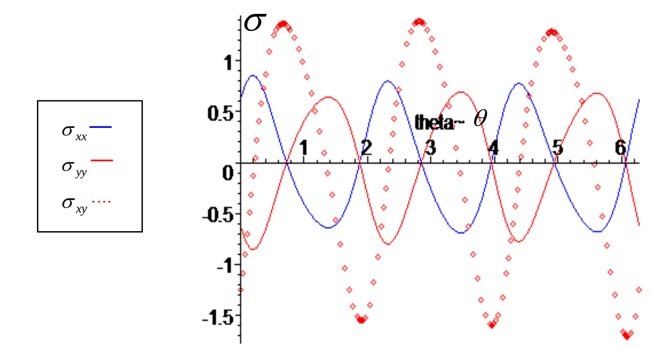
$$\psi(\zeta)=-\frac{w(\zeta^{-1})}{w'(\zeta)}\phi_*(\zeta)+\frac{h\,\zeta^2}{(1-n\,\zeta)^2}\;\phi_*(n^{-1}) \qquad (104)$$

where

$$\phi_*(\zeta)=\phi'(\zeta)-\frac{(X+iY)}{2\pi\,\zeta(1+\chi)}.$$

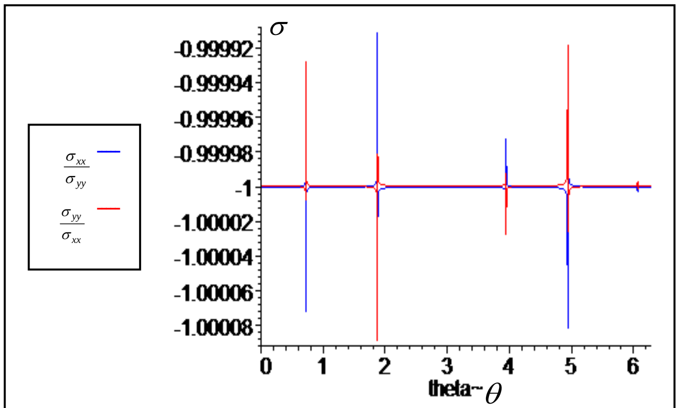
For ($n=0.002$, $m=0.025$, $\ell=0.03$, $c=2$, $x=0.25$, $X=2$, $Y=2$). the stress components σ_{xx}, σ_{yy} and σ_{xy} are obtained in large forms calculated by computer and illustrated in following two cases:

(i) When the study in the normal plate, we have the following shapes for the stress components, see Figs.(12-13)
(ii) In the thetmoelasticity plate, we have the following shapes for the stress components by using the substitutions G=0.5, q=0.1 , r_0=0.75, α =0.7, $\nu = 1$ see Figs.(14-15).



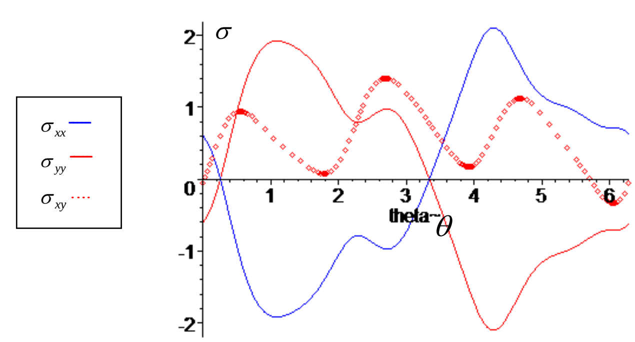
σ_{xx} has positive values at $0 \le \theta \le 0.22611\pi$, $0.58917\pi \le \theta \le 0.89808\pi$ and $1.24841\pi \le \theta \le 1.56687\pi$
σ_{yy} has positive values at $0.22611\ 0 \le \theta \le 0.58917\pi$, $0.89808\pi \le \theta \le 1.24841\pi$ and $1.56687\pi \le \theta \le 1.92038\pi$
σ_{xy} has positive values at $0.06687 \le \theta \le 0.42356\pi$, $0.73885\pi \le \theta \le 1.08917\pi$ and $1.39172\pi \le \theta \le 1.76433\pi$

Fig. 12: The relation between components of stresses and the angle made on the x-axis in the normal plate.



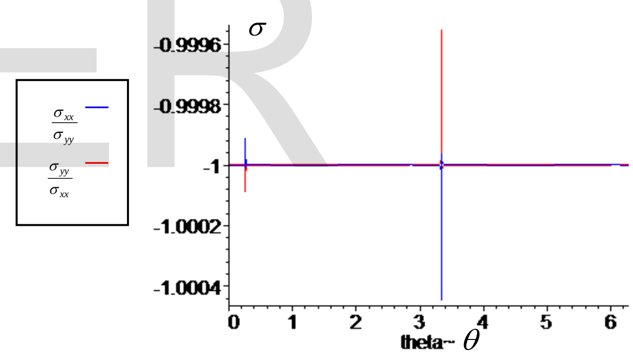
max $\frac{\sigma_{xx}}{\sigma_{yy}}$ at $\theta \approx 0.59235\pi$, max $\frac{\sigma_{yy}}{\sigma_{xx}}$ at $\theta \approx 1.57325\pi$

Fig. 13: The ratio of $\frac{\sigma_{xx}}{\sigma_{yy}}$ and $\frac{\sigma_{yy}}{\sigma_{xx}}$ in the normal plate



σ_{xx} has positive values at $0 \le \theta \le 0.08598\pi$ and $1.0605\pi \le \theta \le 2\pi$, σ_{yy} has positive values at $0.08598\pi \le \theta \le 1.0605\pi$, σ_{xy} has positive values at $0 \le \theta \le 1.8121\pi$

Fig. 14: The relation between components of stresses and the angle made on the x-axis in the thermoelasticity plate.



max $\frac{\sigma_{xx}}{\sigma_{yy}}$ at $\theta \approx 0.0828\pi$, max $\frac{\sigma_{yy}}{\sigma_{xx}}$ at $\theta \approx 1.0636\pi$

Fig. 15 : The ratio of $\frac{\sigma_{xx}}{\sigma_{yy}}$ and $\frac{\sigma_{yy}}{\sigma_{xx}}$ in the thermoelasticity plate.

7 Conclusions

From the previous discussions we have the following results:
1- The solution of the boundary value problem for isotropic homogeneous infinite elastic media in z- plane reduce to obtain the two complex functions, Gaursat functions, by conformal mapping.
2- The conformal mapping $z = w(\zeta)$, $c >$ 0, where $w'(\zeta) \neq 0, \infty$, for $|\zeta|$>1, mapped infinite region to out side a unit circle γ .

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3- Cauchy method is the best method to solving the integro-differential equation with Cauchy kernel and obtaining the two complex functions $\phi(\zeta)$, $\psi(\zeta)$ directly .

4- We find that the effect of heat very clear, we find that the values of components stress are increasing with existence of heat, while at absence of heat we find the values of components stress are reducing .

5- With increasing angle and with absence *f(t)*, we find

$$\left(\frac{\sigma_{xx}}{\sigma_{yy}}\right)^N \text{ and } \left(\frac{\sigma_{yy}}{\sigma_{xx}}\right)^N < \left(\frac{\sigma_{xx}}{\sigma_{yy}}\right)^H \text{ and } \left(\frac{\sigma_{yy}}{\sigma_{xx}}\right)^H$$

6- With increasing angle and with existence *f(t)*, we find

$$\left(\frac{\sigma_{xx}}{\sigma_{yy}}\right)^N < \left(\frac{\sigma_{xx}}{\sigma_{yy}}\right)^H$$

$$\left(\frac{\sigma_{yy}}{\sigma_{xx}}\right)^N > \left(\frac{\sigma_{yy}}{\sigma_{xx}}\right)^H$$

where $\left(\frac{\sigma_{xx}}{\sigma_{yy}}\right)^N$ and $\left(\frac{\sigma_{yy}}{\sigma_{xx}}\right)^N$ in normal state , $\left(\frac{\sigma_{xx}}{\sigma_{yy}}\right)^H$ and $\left(\frac{\sigma_{yy}}{\sigma_{xx}}\right)^H$ after inserting the effect of heating.

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